

Successive Convexification of Non-Convex Optimal Control Problems with State Constraints

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Abstract: This paper presents a successive convexification (SCvx) algorithm to solve a class of non-convex optimal control problems with certain types of state constraints. Sources of non-convexity may include nonlinear dynamics and non-convex state/control constraints. To tackle the challenge posed by non-convexity, first we utilize exact penalty function to handle the nonlinear dynamics. Then the proposed algorithm successively convexifies the problem via a *project-and-linearize* procedure. Thus a finite dimensional convex programming subproblem is solved at each succession, which can be done efficiently with fast Interior Point Method (IPM) solvers. Global convergence to a local optimum is demonstrated with certain convexity assumptions, which are satisfied in a broad range of optimal control problems. As an example, the proposed algorithm is particularly suitable for solving trajectory planning problems with collision avoidance constraints. Through numerical simulations, we demonstrate that the algorithm converges reliably after only a few successions. Thus with powerful IPM based custom solvers, the algorithm can be implemented for real-time autonomous control applications.

Keywords: Optimal control, State constraints, Convex optimization, Successive convexification, Autonomous systems, Trajectory optimization

1. INTRODUCTION

Non-convex optimal control problems emerge in a broad range of science and engineering disciplines. Finding a global solution to these problems is generally considered NP-hard. Using heuristics like simulated annealing, Bertsimas and Nohadani (2010), or combinatorial methods like mixed integer programming, Richards et al. (2002), can compute global optimum solutions for special classes of problems. In many engineering applications, finding a local optimum or even a feasible solution with less computational effort is however a more favorable route. This is particularly the case with real-time control systems, where efficiency and guarantee of convergence are more valuable than optimality. An example in aerospace applications is the planetary landing problem, see Açıkmeşe et al. (2013); Blackmore et al. (2012); Steinfeldt et al. (2010). Non-convexities appeared in this problem include minimum thrust constraints, nonlinear gravity fields and nonlinear aerodynamic forces. State constraints can render the problem non-convex as well. A classic example would be collision avoidance constraints. For instance, Açıkmeşe et al. (2006) discusses the collision avoidance in formation reconfiguration of spacecrafts, Augugliaro et al. (2012) considers the generation of collision-free trajectories for a quadcopter fleet, and Liu and Lu (2014) studies the highly constrained rendezvous problem.

Given the complexity of such non-convex problems, traditional Pontryagin's maximum principle based approaches, e.g. Rockafellar (1972), can fall short. On the other hand,

directly applying optimization methods to solve the discretized optimal control problems gains its popularity thanks to the algorithmic advancement in nonlinear programming, see e.g. Hull (1997); Buskens and Maurer (2000). Unfortunately, general nonlinear optimization can sometimes be intractable in the sense that a bad initial guess could potentially lead to divergence, and also there is few known bounds on the computational effort needed to reach optimality. This makes it difficult to implement for real-time or mission critical applications, because they cannot afford either divergence or a large amount of computational effort. Convex optimization, on the other hand, can be reliably solved in polynomial time to global optimality, see e.g. Boyd and Vandenberghe (2004). More importantly, recent advances have shown that these problems can be solved in real-time by both generic Second Order Cone Programming (SOCP) solvers, e.g. Domahidi et al. (2013), and by customized solvers which take advantage of specific problem structures, e.g. Mattingley and Boyd (2012); Dueri et al. (2014). This motivates researchers to formulate optimal control problems in a convex programming framework for real-time purposes, for example, real-time Model Predictive Control (MPC), see Houska et al. (2011); Zeilinger et al. (2014).

In order to take advantage of these powerful convex programming solvers, one crucial step is to convexify the originally non-convex problems. Recent results on a technique known as *lossless convexification*, e.g. Açıkmeşe and Blackmore (2011); Blackmore et al. (2012); Harris and Açıkmeşe (2014) have proven that certain types of non-

convex control constraints can be posed as convex ones without introducing conservativeness. Liu et al. (2015) also gives a result on convexification of control constraints for the entry guidance application. For nonlinear dynamics and non-convex state constraints, collision avoidance constraints in particular, one simple solution is to perform query-based collision checks, see Allen and Pavone (2016). However, to be mathematically more tractable, Augugliaro et al. (2012); Schulman et al. (2014); Chen et al. (2015) propose to use (variations of) sequential convex programming (SCP) to iteratively convexify them. While these methods usually perform well in practice, no convergence results have been reported yet. To tackle this open challenge, our previous work, Mao et al. (2016), gives a convergence proof of an algorithm that successively convexifies the dynamics with techniques like *virtual control* and *trust regions*.

To include state constraints in our successive convexification (SCvx) algorithmic framework, a few enhancements need to be made. Hauser and Saccon (2006) relaxes the state constraints by using barrier function, but it does not provide theoretical guarantees. In this paper, we first relax the dynamic equations into inequalities by using an exact penalty function, see e.g. Li et al. (2011). Then we propose a *project-and-linearize* procedure to handle state constraints and the relaxed dynamics. While introducing conservativeness is inevitable in this process, the proposed algorithm preserves much more feasibility, compared with similar results in Rosen (1966); Liu and Lu (2014). Finally under some mild assumptions, we present a convergence proof, which not only guarantees that the algorithm will converge, but also demonstrate that the convergent solution recovers the local optimality conditions for the original problem.

One clear advantage of the algorithm proposed in this paper is that this algorithm does not have to resort to trust regions, see e.g. Szmuk and Açıkmeşe (2016); Mao et al. (2016), to guarantee convergence. This property allows the algorithm to potentially take a large step in each succession, thereby greatly accelerating the convergence process, which is exactly the case shown by our numerical simulations. It also worths noting that the proposed algorithm only uses the Jacobian matrix, i.e. the first-order information; therefore we do not have to compute Hessian matrices, i.e. the second-order information, otherwise that task itself could be computationally expensive. To the best of our knowledge, the main contributions of this work are:

- An extended SCvx algorithm with *project-and-linearize* to handle both nonlinear dynamics and non-convex state constraints.
- A convergence proof with local optimality recovery.

2. SUCCESSIVE CONVEXIFICATION

2.1 Problem Formulation

In this paper, we consider the following discrete optimal control problem:

$$\min J(x_i, u_i) := \sum_{i=1}^T \phi(x_i, u_i), \quad (1a)$$

subject to

$$x_{i+1} - x_i = f(x_i, u_i) \quad i = 1, 2, T-1, \quad (1b)$$

$$h(x_i) \geq 0 \quad i = 1, 2, T, \quad (1c)$$

$$u_i \in U_i \subseteq \mathbb{R}^m \quad i = 1, 2, T-1,$$

$$x_i \in X_i \subseteq \mathbb{R}^n \quad i = 1, 2, T.$$

Here x_i, u_i represent discrete state/control at each temporal point. T denotes the final time. X_i, U_i are assumed to be convex and compact sets. We also assume that the objective function in (1a) is continuous and convex, which is the case in many optimal control applications. For example, the minimum fuel problem has $\phi(x_i, u_i) = \|u_i\|$, and the minimum time problem has $\phi(x_i, u_i) = 1$. Equation (1b) represent the system dynamics, where $f(x_i, u_i) \in \mathbb{R}^n$ is, in general, a nonlinear function that is at least twice differentiable. Equation (1c) are the additional state constraints, where $h(x_i) \in \mathbb{R}^s$ is also at least twice differentiable, and could be nonlinear as well. Note that we don't impose non-convex control constraints here because we can leverage the lossless convexification technique, see e.g. Açıkmeşe and Blackmore (2011), to convexify them beforehand. Finally we want to point out it is (1b) and (1c) that render the problem non-convex.

To facilitate our convergence proof, we need a few more assumptions on $f(x_i, u_i)$ and $h(x_i)$. First, we assume each component of $f(x_i, u_i)$ is a convex function over x_i and u_i . In fact, a wide range of optimal control applications, for example systems with double integrator dynamics and aerodynamic drag (constant speed), satisfy this assumption. It also includes all linear systems. Similarly, we assume each component of $h(x_i)$ is convex. An example for this assumption are collision avoidance constraints with the shape of each keep-out zone is convex or can be convexly decomposed. See Figure 1 for a simple illustration. Note that at this stage, these convexity assumptions do not change the non-convex nature of the problem.

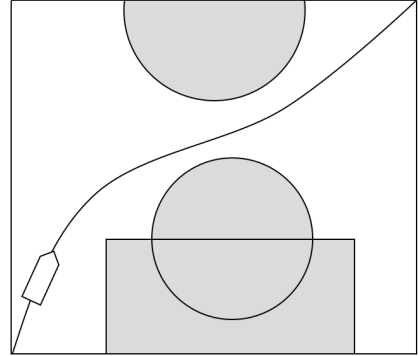


Fig. 1. Convex shaped keep-out zone as state constraints.

To formalize, we give the following hypotheses:

Hypothesis 1. $f_j(x_i, u_i)$ is a convex function over x_i and u_i , $\forall j = 1, 2, \dots, n$.

Hypothesis 2. $h_j(x_i)$ is a convex function over x_i , $\forall j = 1, 2, \dots, s$.

To convert the optimal control problem into a finite dimensional optimization problem, we treat state variable x_i and control variable u_i as a single variable $y = (x_1^T, \dots, x_{T-1}^T, u_1^T, \dots, u_{T-1}^T)^T \in \mathbb{R}^N$, where $N = m(T-1) + nT$. And let Y be the Cartesian product of all X_i and U_i , then $y \in Y$. Y is a convex and compact set because X_i and

U_i are. In addition, we let $g_i(x_i, u_i) = f(x_i, u_i) - x_{i+1} + x_i$, and $g(y) = (g_1^T, g_2^T, \dots, g_{T-1}^T)^T \in \mathbb{R}^{n(T-1)}$. Note that each component of $g(y)$ is convex over y by Hypothesis 1, and the dynamic equation (1b) becomes $g(y) = 0$. We also let $h(y) = (h(x_1)^T, h(x_2)^T, \dots, h(x_T)^T)^T \in \mathbb{R}^{sT}$, then each component of $h(y)$ is convex over y by Hypothesis 2, and (1c) becomes $h(y) \geq 0$. In summary, we have the following non-convex optimization problem:

$$J(y^*) = \min_y \{J(y) | y \in Y, g(y) = 0, h(y) \geq 0\}. \quad (2)$$

By leveraging the exact penalty theory, see e.g. Han and Mangasarian (1979), Mao et al. (2016), we can move $g(y) = 0$ into objective function without compromising optimality:

Theorem 3. (Exactness of penalty function). Let $P(y) = J(y) + \lambda \|g(y)\|_1$ be the exact penalty function, and \bar{y} be a stationary point of

$$\min_y \{P(y) | y \in Y, g(y) \geq 0, h(y) \geq 0\} \quad (3)$$

with Lagrangian multiplier $\bar{\mu}$ for equality constraints. Then if penalty weight λ satisfies $\lambda \geq \|\bar{\mu}\|_\infty$, and if \bar{y} is feasible for (2), then \bar{y} is also a stationary point of (2).

Since each component of $g(y)$ is convex, and $\|\cdot\|_1$ is convex and nondecreasing because we still have constraints $g(y) \geq 0$, by the composition rule of convex function, see Boyd and Vandenberghe (2004), $P(y)$ is a convex function. This marks our first effort towards the convexification of (2).

Let $q(y) = (g(y)^T, h(y)^T)^T \in \mathbb{R}^M$, where $M = sT + n(T-1)$, then we may rewrite (3) as

$$\min_y \{P(y) | y \in Y, q(y) \geq 0\} \quad (4)$$

where $P(y)$ is continuous and convex, and each component of $q(y)$ is convex function. By doing this, we are essentially treating constraints due to dynamics as another keep-out zone. Denote

$$F = \{y | y \in Y, q(y) \geq 0\}$$

as the feasible set. Note that F is compact but not convex.

2.2 Project and Linearize

Since both $f(x_i, u_i)$ and $h(x_i)$ are at least twice differentiable, we have $q(y) \in C^2(Y)$ as well. For any point $z \in F$, let $\nabla_y q_j(z)$ be the Jacobian matrix of $q_j(y)$ evaluated at z .

Now if we directly linearize $q(y)$ at z as in Liu and Lu (2014), since z could be in the interior of F , there will be a gap between the linearized feasible region and F . The gap will increase as z becomes further away from the boundary $q(y) = 0$. This is not a desirable situation, because a fairly large area of the feasible region is not utilized. In other words, we introduced artificial conservativeness. To address this issue, we may introduce a *projection* step, which essentially project z onto each constraint, and get each projection point. Then we linearize each constraint at its own projection point. See Figure 2 for an illustration in \mathbb{R}^N .

To formalize, let $q_j(y), j = 1, 2, \dots, M$ represent each component of $q(y)$, i.e. each constraint. Note that $q_j(y)$ is a convex function, hence $q_j(y) \leq 0$ is a closed convex set. Using the well-known Hilbert projection theorem, see e.g. Wulbert (1968), we have

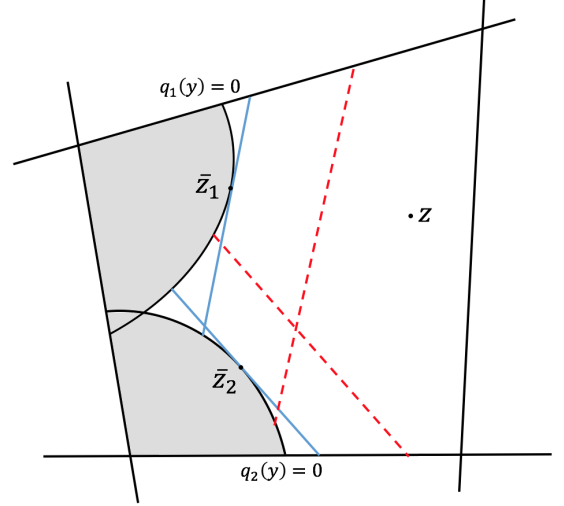


Fig. 2. Project-and-linearize vs directly linearize.

Theorem 4. (Uniqueness of projection). For any $z \in F$, there exists a unique point

$$\bar{z} = \underset{y}{\operatorname{argmin}} \{ \|z - y\|_2 | q_j(y) \leq 0 \}, \quad (5)$$

called the projection of z on $q_j(y) \leq 0$.

Equation (5) is a simple convex program with small dimensions. It can be solved very fast, e.g., in sub-milliseconds, using any convex programming solver, so the additional computation is not expensive.

Doing this for each constraint, we will get a set of projection points, $\{\bar{z}_1, \bar{z}_2, \dots, \bar{z}_M\}$. Note that these projection points must lie on the boundary of $q_j(y) \leq 0$, i.e.

$$q_j(\bar{z}_j) = 0, \quad \forall j = 1, \dots, M$$

For a fixed $z \in F$, let $l_j(y, z)$ be the linear approximation of $q_j(y)$:

$$l_j(y, z) = \nabla_y q_j(\bar{z}_j)(y - \bar{z}_j), \quad (6)$$

and let $l(y, z) = (l_1^T, l_2^T, \dots, l_M^T)^T \in \mathbb{R}^M$. For each $z \in F$, denote

$$F_z = \{y | y \in Y, l(y, z) \geq 0\}$$

as the feasible region after linearization. F_z also defines a point-to-set mapping, $F_z : z \rightarrow F_z$. Note that each component of $l(y, z) \geq 0$, i.e. $l_j(y, z) \geq 0$ represents a half-space. Hence $l(y, z) \geq 0$ is the intersection of half-spaces, which means F_z is a convex and compact set.

Remark. Convexification of F by using F_z also inevitably introduces conservativeness, but one can verify that it is the best we can do to maximize feasibility while preserving convexity.

The following lemma gives an invariance result regarding the point-to-set mapping F_z . It is essential to our subsequent analyses.

Lemma 5. (Invariance of F_z). For each $z \in F$,

$$z \in F_z \subseteq F.$$

Proof. For each $z \in F$, and $\forall j = 1, 2, \dots, M$, from (6), we have

$$l_j(z, z) = \nabla_y q_j(\bar{z}_j)(z - \bar{z}_j).$$

Since \bar{z}_j is the projection, $(z - \bar{z}_j)$ is the normal vector at \bar{z}_j , which is aligned with the gradient $\nabla_y q_j(\bar{z}_j)$. Hence

$\nabla_y q_j(\bar{z}_j)(z - \bar{z}_j) \geq 0$, i.e. $l_j(z, z) = 0, \forall j = 1, 2, \dots, M$, i.e. $z \in F_z$.

Furthermore, since $q_j(y)$ is a convex function, we have for any $y \in F_z$,

$$q_j(y) \geq q_j(\bar{z}_j) + \nabla_y q_j(\bar{z}_j)(y - \bar{z}_j) = l_j(y, z) \geq 0,$$

which means $y \in F$. Hence $F_z \subseteq F$. \square

2.3 The SCvx Algorithm

Now we have a convex and compact feasible region F_z and a convex objective function $P(y)$, we are ready to present a successive procedure to solve the non-convex problem in (4). Note that the feasible region F_z is defined by z . Therefore, if we start from a point $z^{(0)} \in F$, a sequence $\{z^{(k)}\}$ will be generated as in each iteration,

$$z^{(k+1)} = \underset{y}{\operatorname{argmin}} \{P(y) | y \in F_{z^{(k)}}\}, \quad k = 0, 1, \dots \quad (7)$$

This is a convex programming sub-problem, whose global minimizer is attained at $z^{(k+1)}$. At these intermediate steps, $z^{(k+1)}$ may not be the optimal solution to (4). Our goal, however, is to prove that this sequence $\{z^{(k)}\}$ converges to a limit point z^* , and this limit point solves (4) by *project-and-linearize* at z^* itself, i.e., it is a “fixed-point” satisfying

$$z^* = \underset{y}{\operatorname{argmin}} \{P(y) | y \in F_{z^*}\}. \quad (8)$$

More importantly, we want to show that z^* gives a local optimum to (4) convexified at z^* itself. Then by solving a sequence of convex programming subproblems, we effectively solved the non-convex optimal control problem in (2) because of Fact 3. Therefore we call this procedure the Successive Convexification (SCvx) algorithm. It is summarized in Algorithm 1.

Algorithm 1 Successive Convexification Algorithm

- 1: **procedure** SCvx($z^{(0)}, \lambda$)
 - 2: **input** Initial point $z^{(0)} \in F$. Penalty weight $\lambda \geq 0$.
 - 3: **while** not converged **do**
 - 4: **step 1** At each succession k , we have the current point $z^{(k)}$. For each constraint $q_j(y)$, solve the problem in (5) to get a projection point $\bar{z}_j^{(k)}$.
 - 5: **step 2** Construct the convex feasible region $F_{z^{(k)}}$ by using (6).
 - 6: **step 3** Solve the convex subproblem in (7) to get $z^{(k+1)}$. Let $z^{(k)} \leftarrow z^{(k+1)}$ and go to the next succession.
 - 7: **end while**
 - 8: **return** $z^{(k+1)}$.
 - 9: **end procedure**
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3. CONVERGENCE ANALYSIS

In this section, we proceed to show that Algorithm 1 does converge to a point z^* that indeed satisfies (8). First we must assume the application of regular constraint qualifications, namely the Linear Independence Constraint Qualification (LICQ) and the Slater’s condition. They can be formalized as the following:

Hypothesis 6. (LICQ). For each $z \in F$, the Jacobian matrix $\nabla_y q(z)$ has full row rank, i.e. $\operatorname{rank}(\nabla_y q(z)) = M$.

Hypothesis 7. (Slater’s condition). For each $z \in F$, the convex feasible region F_z contains interior points.

These assumptions do impose some practical restrictions on our feasible region. Figure 3 shows some examples where LICQ or Slater’s condition might fail. For scenarios like (a), we may perturb our discrete points to break symmetry. For scenarios like (b), we have to assume the connectivity of the feasible region. In other words, our feasible region cannot be degenerate at some point, for example, in collision avoidance the feasible region is not allowed to be completely obstructed.

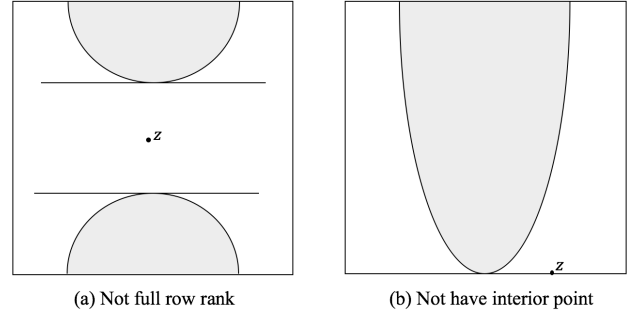


Fig. 3. Cases where LICQ or Slater’s condition fails

In order to analyze the convergence, first we need to show that the point-to-set mapping F_z is continuous in the sense that given any point $z^{(1)} \in F$ and $y^{(1)} \in F_{z^{(1)}}$, then for any point $z^{(2)} \in F$ in the neighborhood of $z^{(1)}$, there exists a $y^{(2)} \in F_{z^{(2)}}$ that is close enough to $y^{(1)}$.

First, We have the following lemma:

Lemma 8. (Lipschitz continuity of $l(y, z)$). The linear approximation $l(y, z)$ in (6) is Lipschitz continuous in z , that is

$$\|l(y, z^{(1)}) - l(y, z^{(2)})\| \leq \gamma \|z^{(1)} - z^{(2)}\|, \quad (9)$$

for any $z^{(1)}, z^{(2)} \in F$ and constant γ .

Proof. Each $l_j(y, z)$ is in fact a composition of two mappings. The first mapping maps z to its projection \bar{z}_j . This mapping is defined by the optimization problem in (5). It is a well known result that this mapping is non-expansive (i.e. Lipschitz continuous with constant 1), See e.g. Wulbert (1968). The second mapping is defined by the auxiliary function:

$$a_j(y, \bar{z}_j) = \nabla_y q_j(\bar{z}_j)(y - \bar{z}_j),$$

Since $q(y) \in C^2(Y)$, $a_j(y, \bar{z}_j)$ is Lipschitz continuous in \bar{z}_j . By the composition rule of two Lipschitz continuous functions, we have $l_j(y, z)$ is Lipschitz continuous, for all $j = 1, 2, \dots, M$. Therefore, sum over j gives the Lipschitz continuity of $l(y, z)$. \square

Now with Hypothesis 6, 7 and Lemma 8, we are ready to prove the continuity of point-to-set mapping F_z . The result is given as follows:

Lemma 9. (Continuity of mapping F_z). Given $z^{(1)} \in F$ and $y^{(1)} \in F_{z^{(1)}}$, then given $\epsilon > 0$, there exists a $\delta > 0$ so that for any point $z^{(2)} \in F$ with $\|z^{(2)} - z^{(1)}\| \leq \delta$, there exists a $y^{(2)} \in F_{z^{(2)}}$ such that $\|y^{(2)} - y^{(1)}\| \leq \epsilon$.

Proof. From Hypothesis 6, we know that the Jacobian matrix $\nabla_y q(z)$ has full row rank. Thus matrix

$\nabla_y q(z) \nabla_y q(z)^T$ is symmetric and positive definite for any $z \in F$. Consequently, there exists β such that

$$\|(\nabla_y q(z) \nabla_y q(z)^T)^{-1}\| \leq \beta^2, \quad \forall z \in F. \quad (10)$$

If $y^{(1)} \in F_{z^{(2)}}$, then take $y^{(2)} = y^{(1)}$ such that $\|y^{(2)} - y^{(1)}\| = 0 \leq \epsilon$. Now suppose $y^{(1)} \notin F_{z^{(2)}}$, then there exists at least one j , such that $l_j(y^{(1)}, z^{(2)}) < 0$. Let

$$\bar{l}_j = \begin{cases} l_j(y^{(1)}, z^{(2)}) & l_j(y^{(1)}, z^{(2)}) < 0, \\ 0 & l_j(y^{(1)}, z^{(2)}) \geq 0. \end{cases}$$

Note that $l_j(y^{(1)}, z^{(1)}) \geq 0$, then by definition,

$$|\bar{l}_j| \leq |l_j(y^{(1)}, z^{(1)}) - l_j(y^{(1)}, z^{(2)})|, \quad \forall j = 1, 2, \dots, M.$$

Hence we have

$$\|\bar{l}\| \leq \|l(y^{(1)}, z^{(1)}) - l(y^{(1)}, z^{(2)})\| \leq \gamma \|z^{(1)} - z^{(2)}\|. \quad (11)$$

The second inequality follows from (9) in Lemma 8.

Now we consider two cases:

Case 1: $y^{(1)}$ is an interior point of Y , then $\exists \epsilon_1 \in (0, \epsilon]$, such that $y \in Y$ for all y satisfies $\|y - y^{(1)}\| \leq \epsilon_1$.

Let $\|z^{(2)} - z^{(1)}\| \leq \delta$, with $\delta = \epsilon_1 / \beta \gamma$, and let

$$y^{(2)} = y^{(1)} - \nabla_y q(z^{(2)})^T (\nabla_y q(z^{(2)}) \nabla_y q(z^{(2)})^T)^{-1} \bar{l}. \quad (12)$$

We need to verify that $y^{(2)} \in F_{z^{(2)}}$. First, we have

$$l_j(y^{(2)}, z^{(2)}) = \nabla_y q_j(\bar{z}_j^{(2)})(y^{(2)} - \bar{z}_j^{(2)}). \quad (13)$$

Substitute $y^{(2)}$ from (12) into the stacked form of (13), and then unstack, we get

$$\begin{aligned} l_j(y^{(2)}, z^{(2)}) &= \nabla_y q_j(\bar{z}_j^{(2)})(y^{(1)} - \bar{z}_j^{(2)}) - \bar{l}_j \\ &= l_j(y^{(1)}, z^{(2)}) - \bar{l}_j \\ &\geq 0. \end{aligned}$$

The last inequality follows from the definition of \bar{l}_j . Therefore, $l(y^{(2)}, z^{(2)}) \geq 0$.

Next we need to show that $y^{(2)} \in Y$. Rearrange terms in (12), we have

$$\begin{aligned} \|y^{(2)} - y^{(1)}\|^2 &= \bar{l}^T (\nabla_y q(z^{(2)}) \nabla_y q(z^{(2)})^T)^{-1} \bar{l} \\ &\leq \beta^2 \|\bar{l}\|^2 \\ &\leq \beta^2 \gamma^2 \|z^{(1)} - z^{(2)}\|^2 \end{aligned}$$

The inequalities follow from (10) and (11). Thus we have $\|y^{(2)} - y^{(1)}\| \leq \beta \gamma \delta = \epsilon_1$, i.e. $y^{(2)} \in Y$. So now we have verified $y^{(2)} \in F_{z^{(2)}}$. Since $\epsilon_1 \leq \epsilon$, we have $\|y^{(2)} - y^{(1)}\| \leq \epsilon$, so that F_z is continuous.

Case 2: $y^{(1)}$ is a boundary point of Y . From Hypothesis 7, $F_{z^{(1)}}$ has interior points. Also since $F_{z^{(1)}}$ is a convex set, it is a well known fact that there are interior points in the ϵ -Neighborhood of every point in $F_{z^{(1)}}$. Then we can apply the same argument as in Case 1, to get that F_z is continuous. \square

One way to show convergence is to demonstrate the convergence of objective functions, $P(z^{(k)})$. Now let's define

$$\Phi(z) := \min_y \{P(y) | y \in F_z\}$$

to be the function that maps the point z we are solving at in each iteration to the optimal value of the objective function in F_z . By using the continuity of the point-to-set mapping F_z , the following lemma gives the continuity of the function $\Phi(z)$.

Lemma 10. $\Phi(z)$ is continuous for $z \in F$.

Proof. Given any two points $z^{(1)}, z^{(2)} \in F$, and they are close to each other, i.e. $\|z^{(1)} - z^{(2)}\| \leq \delta$. Let

$$y^{(1)} = \underset{y}{\operatorname{argmin}} \{P(y) | y \in F_{z^{(1)}}\} \text{ and}$$

$$y^{(2)} = \underset{y}{\operatorname{argmin}} \{P(y) | y \in F_{z^{(2)}}\},$$

then we have $\Phi(z^{(1)}) = P(y^{(1)})$ and $\Phi(z^{(2)}) = P(y^{(2)})$.

Without loss of generality, let $\Phi(z^{(1)}) \leq \Phi(z^{(2)})$, i.e. $P(y^{(1)}) \leq P(y^{(2)})$. From Lemma 9, the continuity of F_z , there exists $\hat{y}^{(2)} \in F_{z^{(2)}}$, such that $\|\hat{y}^{(2)} - y^{(1)}\| \leq \eta$ for any $\eta > 0$. Then since $P(y)$ is continuous, we have

$$|P(\hat{y}^{(2)}) - P(y^{(1)})| \leq \epsilon. \quad (14)$$

Since $y^{(2)}$ is the minimizer of $P(y)$ in $F_{z^{(2)}}$, $P(y^{(2)}) \leq P(\hat{y}^{(2)})$. Therefore, $P(y^{(1)}) \leq P(\hat{y}^{(2)})$ by assumption. Now (14) becomes $P(\hat{y}^{(2)}) - P(y^{(1)}) \leq \epsilon$. Again, because $P(y^{(2)}) \leq P(\hat{y}^{(2)})$, we have $P(y^{(2)}) - P(y^{(1)}) \leq \epsilon$, i.e. $\Phi(z^{(2)}) - \Phi(z^{(1)}) \leq \epsilon$, which means $\Phi(z)$ is continuous. \square

With the continuity of $\Phi(z)$ in hand, now we are ready to present our final convergence results:

Theorem 11. (Global convergence). The sequence $\{z^{(k)}\}$ generated by the successive procedure (7) is in F , and has limit point z^* , at which the corresponding sequence $\{P(z^{(k)})\}$ attains its minimum, $P(z^*)$. More importantly, $P(z^*) = \Phi(z^*)$, i.e. z^* is a local optimum of the penalty problem in (4) convexified at z^* .

Proof. From Lemma 5, we have $z^{(k)} \in F_{z^{(k)}} \subseteq F$, then

$$P(z^{(k+1)}) = \min_y \{P(y) | y \in F_{z^{(k)}}\} \leq P(z^{(k)}),$$

because $z^{(k)}$ is a feasible point to this convex optimization problem, while $z^{(k+1)}$ is the optimum. Therefore, the sequence $\{P(z^{(k)})\}$ is monotonically decreasing.

Also since $F_{z^{(k)}} \subseteq F$ for all k , we have

$$P(z^{(k+1)}) \geq \min_y \{P(y) | y \in F\},$$

which means the sequence $\{P(z^{(k)})\}$ is bounded from below. Then by the monotone convergence theorem, see e.g. Rudin (1964), $\{P(z^{(k)})\}$ converges to its infimum. Due to the compactness of $F_{z^{(k)}}$, this infimum is attained by all the convergent subsequences of $\{z^{(k)}\}$. Let $z^* \in F$ be one of the limit points, then $\{P(z^{(k)})\}$ attains its minimum at $P(z^*)$, i.e.

$$P(z^*) \leq P(z^{(k)}), \quad \forall k = 0, 1, \dots \quad (15)$$

To show $P(z^*) = \Phi(z^*)$, we note that since $z^* \in F$, $z^* \in F_{z^*}$ by Lemma 5. Thus we have

$$\Phi(z^*) := \min_y \{P(y) | y \in F_{z^*}\} \leq P(z^*).$$

Now for the sake of contradiction, we suppose $\Phi(z^*) < P(z^*)$. Then by Lemma 10, the continuity of $\Phi(z)$, there exists a sufficiently large k such that $\Phi(z^k) < P(z^*)$. Then we have $P(z^{(k+1)}) = \Phi(z^k) < P(z^*)$, which contradicts (15). Therefore $P(z^*) = \Phi(z^*)$ holds. \square

4. NUMERICAL RESULTS

In this section, we present numerical results that apply the SCvx algorithm to an aerospace problem. Consider a small

Table 1. Parameter Values

Par.	Value	Par.	Value
N	17	ϵ	1×10^{-6}
V_{\max}	2 m/s	u_{\max}	13.33 m/s ²
g	$[0, 0, -9.81]^T$ m/s ²	θ_{cone}	30 deg
p_0	$[-8, -1, 0]^T$ m	p_f	$[8, 1, 0.5]^T$ m
v_0	$[0, 0, 0]^T$ m/s	v_f	$[0, 0, 0]^T$ m/s
$p_{c,1}$	$[-1, 0]^T$ m	$p_{c,2}$	$[4, -1]^T$ m
r_1	3 m	r_2	1.5 m
λ	0	\hat{n}	$[0, 0, 1]^T$
t_f	15 s		

multi-rotor vehicle with state at time t_i

$$x_i = \begin{bmatrix} p_i \\ v_i \end{bmatrix}$$

where $p_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ represent vehicle position and velocity at time t_i respectively. We assume that vehicle motion is adequately modeled by double integrator dynamics with constant time step Δt such that $x_{i+1} = Ax_i + B(u_i + g)$, where $u_i \in \mathbb{R}^3$ is the control at time t_i , $g \in \mathbb{R}^3$ is a constant gravity vector,

$$A = \begin{bmatrix} I_{3 \times 3} & \Delta t \cdot I_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \text{ and, } B = \begin{bmatrix} \frac{1}{2} \Delta t^2 \cdot I_{3 \times 3} \\ \Delta t \cdot I_{3 \times 3} \end{bmatrix}.$$

Above, $I_{3 \times 3}$ is the 3×3 multiplicative identity matrix and $0_{3 \times 3}$ is the 3×3 additive identity (or zero) matrix. Further, we impose a speed upper-bound at each time step, $\|v_i\|_2 \leq V_{\max}$, an acceleration upper-bound such that $\|u_i\|_2 \leq u_{\max}$ (driven by a thrust upper-bound), and a thrust cone constraint $\hat{n}^T u_i \geq \|u_i\|_2 \cos(\theta_{\text{cone}})$ that constrains the thrust vector to a cone pointing towards the unit vector \hat{n} (pointing towards the ceiling) with angle θ_{cone} . Finally, the multi-rotor must avoid a known set of cylindrical obstacles with the j^{th} cylinder having center $p_{c,j} \in \mathbb{R}^2$ and radius $r_j \in \mathbb{R}_+$. Therefore, each state must satisfy the non-convex constraint given by $\|Hx_i - p_{c,j}\|_2 \geq r_j$ where H is a linear mapping that maps p_i to its projection on the ground plane.

Given these constraints, the objective is to find a minimum fuel trajectory from a prescribed $x(t_0)$ to a known $x(t_f)$ with fixed final time t_f and N discrete points along with n_{cyl} cylindrical obstacles to avoid:

$$\begin{aligned} \min \quad & \sum_{i=1}^N \|u_i\| \\ \text{s.t.} \quad & x_{i+1} = Ax_i + B(u_i + g), \quad i = 1, \dots, N-1 \\ & \|u_i\|_2 \leq u_{\max}, \quad i = 1, \dots, N, \\ & \|v_i\|_2 \leq V_{\max}, \quad i = 1, \dots, N, \\ & \hat{n}^T u_i \geq \|u_i\|_2 \cos(\theta_{\text{cone}}), \quad i = 1, \dots, N, \\ & \|Hx_i - p_{c,j}\|_2 \geq r_j, \quad i = 1, \dots, N, \quad j = 1, \dots, n_{\text{cyl}}, \\ & x_0 = x(t_0), \quad x_N = x(t_f). \end{aligned}$$

The parameters given in Table 1 are used to obtain the numerical results presented herein. An initial feasible trajectory is obtained by using the Trust Region Method (TRM) given in Mao et al. (2016) to solve the feasibility problem associated with this optimization. We have found that a feasible solution can generally be found with less than three iterations, where each iteration solves an SOCP. The

ground plane projection of the feasible initial trajectory, $z^{\{0\}}$ is shown in Figure 4 (black circles). Note that $z^{\{0\}}$ is not only feasible for the non-convex state constraints, but also for the convex constraints imposed on the trajectory.

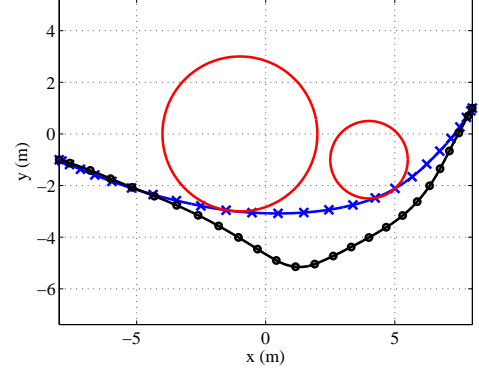


Fig. 4. 2-D Ground plane projection of initial trajectory (black circles) and the converged trajectory (blue x's).

The **SCvx** algorithm is initiated with $z^{\{0\}}$, and is considered to have converged when the improvement in the cost of the linearized problem is less than ϵ . Figure 5 illustrates the converged trajectory that avoids the cylindrical obstacles while satisfying its actuator and mission constraints (blue x's). Note that the ground plane projection of the converged trajectory $z^{\{5\}}$ in Figure 4 (blue x's) is different than that of $z^{\{0\}}$ and is characterized by having a smooth curve. For $z^{\{0\}}$, $\sum_{i=1}^N \|u_i\| = 269.01$ and at $z^{\{5\}}$, we have that $\sum_{i=1}^N \|u_i\| = 245.38$, so the cost of the converged trajectory is lower than that of the initial trajectory. At each iteration, the **SCvx** algorithm solves an SOCP, and therefore 5 SOCPs were solved in order to produce these results (in addition to 2 SOCPs for finding a feasible starting point).

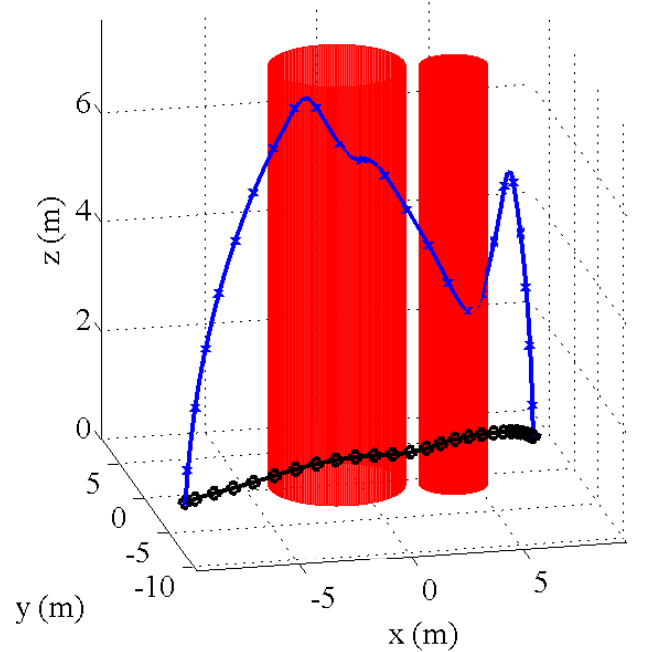


Fig. 5. 3-D plot of the initial trajectory (black circles) and the converged trajectory (blue x's).

For comparison, we solve the same problem using only the TRM given in Mao et al. (2016). The converged trajectories for both algorithms are identical to within ϵ . The number of iterations (k_{TRM} and k_{SCvx}) and time elapsed using each method (t_{TRM} and t_{SCvx}) are reported in Table 2. The 2 iterations necessary to find a feasible starting point for the SCvx algorithm are also reported. All times were found using the ECOS solver (Domahidi et al. (2013)) on a standard workstation with an Intel Xeon processor at 3.40 GHz and 16 GB of RAM.

Table 2. Runtimes and Iterations

Method	k_{TRM}	t_{TRM}	k_{SCvx}	t_{SCvx}	k_{tot}	t_{tot}
TRM	14	149.9 ms	-	-	14	149.9 ms
SCvx	2	39.3 ms	5	26.5 ms	7	65.7 ms

5. CONCLUSION

As the paper shows, the proposed successive convexification (SCvx) algorithm with *project-and-linearize* procedure can solve a class of non-convex optimal control problems by solving a sequence of convex optimization problems. More importantly, we give a convergence proof demonstrating that the algorithm will always converge to a local optimum. Numerical results suggest that this convergence process only takes a very few number of iterations, which is superior to most trust-region-method based algorithms. Furthermore, the runtimes we obtained place the method in the realm of real-time applications.

This algorithm obviously has its limitations, for one it needs a feasible point to start with, so a lifting procedure such that the algorithm can take any initial guess would be a great enhancement. Another theoretical refinement would be getting rid of the differentiability assumptions by using subgradient theories. This will enable the algorithm to handle non-smooth constraints like polyhedra shaped obstacles.

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